Homework Question: Show $\binom{2n}{n} \geq \frac{4^n}{2n} = \frac{2^{2n}}{2n+1}$

Aside: Pascal's triangle

$(A+B)^n$ coeffs

$(1+1)^n = 2^n$

$$\begin{array}{ll}
 & 1 \\
n=1 & 1 \quad 1 \\
n=2 & 1 \quad 2 \quad 1 \\
n=3 & 1 \quad 3 \quad 3 \quad 1 \longrightarrow \text{sum} = 2^n \\
n=4 & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
 & 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}$$

$\therefore \frac{4^n}{2n} =$ average of the numbers in the row, if we combine the first & last numbers (1 & 1) to make 2.

largest # in row $= \binom{2n}{n} \geq$ avg.

$$\Rightarrow \boxed{\binom{2n}{n} \geq \frac{4^n}{2n}}.$$

Q: Find the number $U(n)$ of bit strings of length n that do not have consecutive zeros.
(eg 01101 is a bit string of length 5).

Let's play: $n=1$ 0 or 1 $U(1)=2$

$n=2$ 01, 10, 11 $U(2)=3$

$n=3$ 010, 011, 101, 110, 111 $U(3)=5$

Let's try to make a recursive formula:

If we $U(1), \dots, U(n)$,

$$U(n+1) = U(n) + U(n-1)$$

$$\Big(\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{n}\ 1\Big)$$

$U(n)$ possibilities

no conseq zeros

1 allowed

$$\underbrace{\_\ \_\ \_\ \_\ \_}_{n-1}\ 1\ \ 0 \text{ or}$$

no conseq zeros

$$\Rightarrow U(3) = U(2) + U(1) = 2 + 3 = 5. ✓$$

$$U(4) = U(3) + U(2) = 5 + 3 = 8$$

$$U(5) = U(4) + U(3) = 8 + 5 = 13.$$

In fact $U(n) = F_{n+2}$, where

F_k is the k^{th} Fibonacci number.

Is there a formula for

$\text{Fact}(n) = n!$?

Yes: $$\boxed{n! = \int_0^\infty t^n e^{-t}\, dt.}$$

Gamma function $$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

$\Gamma(p) = p-1$ factorial

integer $\quad (p-1)!$

$\boxed{\text{Example}}$ Find the number of ternary strings (0, 1, 2) of length n that do not contain 00 or 22.

eg length 4: 2101, 2111

Call this $T(n)$ = # of ternary strings without 00 or 22.

$$T(1) = (0 \text{ or } 1 \text{ or } 2) = 3$$

$$T(2) = 7$$

01, 02, 10, 11, 12, 20, 21

Let's try to find a recursive formula for $T(n+1)$ in terms of $T(n)$, etc.

$$T(n+1) = ?$$

$$\underbrace{\underline{\qquad\qquad\qquad\qquad}}_{n \text{ digits}} \ \underline{\quad}$$

$T(n)$ possibilities

Could be

1 no matter what

0 if previous # is not 0

2 if previous # is not 2

This means, for each of the $T(n)$ possibilities for the first n digits, there are at least 2 possible last digits.

In the case where the n^{th} digit is a 1, there are 3 possible n^{th} digits.

The number of cases where the n^{th} digit is 1 is $T(n-1)$ (or 1 if $n=1$)

So $T(n+1) = 2T(n) + \underbrace{T(n-1)}_{1 \text{ if } n=1}$.

We check $T(2) = 2T(1) + 1 = 2(3)+1 = 7$

$T(3) = 2T(2) + T(1) = 2(7)+3$
$= 17.$

Check: 010, 011, 012, 020, 021, 101, 102, 110, 111, 112, 120, 121, 201, 202, 210, 211, 212.

17 possibilities!
