

Homework Question: Show $\binom{2n}{n} \geq \frac{4^n}{2n+1} = \frac{2^{2n}}{2n+1}$

Aside: Pascal's triangle

$$(A+B)^n \text{ coeffs}$$

$$(1+1)^n = 2^n$$

$$\begin{array}{c} n=1 \quad 1 \quad 1 \\ n=2 \quad 1 \quad 2 \quad 1 \\ n=3 \quad 1 \quad 3 \quad 3 \quad 1 \rightarrow \text{sum} = 2^n \\ n=4 \quad 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ \quad \quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

$\therefore \frac{4^n}{2n+1}$ is average of the numbers in the row, if we combine the first & last numbers (1 & 1) to make 2.

largest # in row
 $= \binom{2n}{n} \geq \text{avg.}$

$$\Rightarrow \boxed{\binom{2n}{n} \geq \frac{4^n}{2n+1}}$$

Q: Find the number $U(n)$ of bit strings of length n that do not have consecutive zeros.
(eg 01101 is a bit string of length 5).

Let's play: $n=1$ 0 or 1 $U(1)=2$

$n=2$ 01, 10, 11 $U(2)=3$

$n=3$ 010, 011, 101, 110, 111 $U(3)=5$

Let's try to make a recursive formula:

If we $U(1), \dots, U(n)$,

$$U(n+1) = U(n) + U(n-1)$$

$U(n)$ possibilities $\rightarrow$  $\uparrow$ no consec zeros 1 allowed

$$\begin{aligned} \Rightarrow U(3) &= U(2) + U(1) \\ &= 2 + 3 \\ &= 5. \checkmark \end{aligned}$$

 $\downarrow$ 0 or 1 no consec zeros

$$\begin{aligned} U(4) &= U(3) + U(2) \\ &= 5 + 3 = 8 \end{aligned}$$

$$U(5) = U(4) + U(3) = 8 + 5 = 13.$$

In fact $U(n) = F_{n+2}$, where

F_k is the k^{th} Fibonacci number.

Is there a formula for

$$\text{Fact}(n) = n! \quad ?$$

Yes: $n! = \int_0^{\infty} t^n e^{-t} dt$

Gamma
function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

$$\Gamma(p) = p-1 \text{ factorial}$$

$\uparrow$
integer $(p-1)!$

Example Find the number of ^{0,1,2}ternary strings of length n that do not contain 00 or 22.

eg length 4 : 2101, 2111

Call this $T(n)$ = # of ternary strings
without 00 or 22.

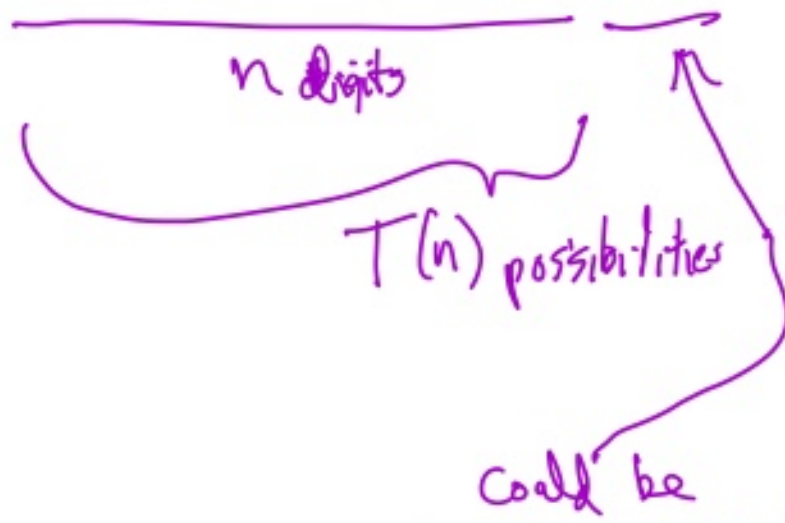
$$T(1) = (0 \text{ or } 1 \text{ or } 2) = 3$$

$$T(2) = 7$$

01, 02, 10, 11, 12, 20, 21

Let's try to find a recursive formula
for $T(n+1)$ in terms of $T(n)$, etc.

$$T(n+1) = ?$$



1 no matter what
0 if previous # is not 0
2 if previous # is not 2

This means, for each of the $T(n)$ possibilities for

the first n digits, there are at least 2 possible last digits.

In the case where the p_2^{th} digit is a 1, there are 3 possible n^{th} digits.

The number of cases where the n^{th} digit is 1 is $\xrightarrow{n-1} 1 \cdot T(n-1)$ (or 1 if $n=1$)

$$\text{So } T(n+1) = 2T(n) + \underbrace{T(n-1)}_{1 \text{ if } n=1}$$

$$\text{We check } T(2) = 2T(1) + 1 = 2(3) + 1 = 7$$

$$T(3) = 2T(2) + T(1) = 2(7) + 3 = 17.$$

Check: 010, 011, 012, 020, 021, 022, 101, 102, 110, 111, 112, 120, 121, 201, 202, 210, 211, 212.

(7 possibilities!)
